LARGE SCALE BEHAVIOR OF SYSTEMS OF REFLECTING DIFFUSIONS WITH LOCAL-TIME DEPENDENT DRIFT

Abstract. We study systems of diffusions reflecting inside a sufficiently smooth domain $D \subset \mathbb{R}^d$ whose drift depends on the reflection local time. The one particle was studied by Bass, Burdzy, Chen, and Hairer (2010). We show that a strong exchangeability property together with strong uniqueness of the limit, implies a strong version of propagation of chaos. The collection of empirical processes converge to a non-linear reflecting heat equation whose drift depends globally on the boundary. Furthermore, existence and uniqueness of this PDE are demonstrated using stochastic methods.

1. Introduction

We study systems of diffusions reflecting inside a domain $D \subset \mathbb{R}^d$, with C^2 smooth boundary, whose drift depends on the reflection local time. The case of one particle was studied in varying degrees of generality, by Knight [Kni01], White [Whi07], and Bass, Burdzy, Chen and Hairer [BBCH10].

2. System of Diffusions with Inert Drift

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be probability space supporting n independent $\mathcal{F}_t$-adapted $\mathbb{R}^d$ Brownian motions $\{B_i(t) : t \in [0,T], 1 \leq i \leq n\}$. To ensure our processes are adapted we assume $\mathcal{F}_0$ also contain $\sigma(X_i(0) : 1 \leq i \leq n)$ where $X_i(0)$ are i.i.d. random variables with values inside D. For each $x \in \partial D$, let $\eta(x) \in \mathbb{R}^d$ be the unit inward normal of ∂D at x. Given this setup, we consider a system of processes $(X_i^{(n)}, L_i^{(n)}), 1 \leq i \leq n$ such that almost surely:

{intro:SDE_system}

$$
\begin{aligned}
X_i^{(n)}(t) &:= X_i(0) + B_i(t) + L_i^{(n)}(t) + \int_0^t K^{(n)}(s)\,\mathrm{d}s \in D,\\
K^{(n)}(t) &= \frac{1}{n}\int_0^t \sum_{i=1}^n \eta\big(X_i^{(n)}(s)\big)\,\mathrm{d}|L_i^{(n)}|(s), \qquad (1)\\
L_i^{(n)}(t) &= \int_0^t \eta\big(X_i^{(n)}(s)\big)\,\mathrm{d}|L_i^{(n)}|(s),
\end{aligned}
$$

for all $t \in [0,T]$, where $t \to |L_i^{(n)}|(t)$ is the continuous, non-decreasing, local-time of $X_i^{(n)}$ on ∂D. See [BBCH10] for more discussion on the boundary local-time.

{lemma:s_map}

lemma:s_map **Lemma 2.1** (Skorohod Map of Reflection, [LS84]). *For any $f \in C([0,T],\mathbb{R}^d)$ with $f(0) \in D$, there is a pair of continuous functions (x_f, ℓ_f) such that*

eq:smap1 (2) {eq:smap1}
$$x_f(t) = f(t) + \ell_f(t) \in D, \tag{2}$$

eq:smap2 (3) {eq:smap2}
$$\ell_f(t) = \int_0^t \eta(x_f(s))\mathrm{d}|\ell_f|(s) \tag{3}$$

where $t \to |\ell_f|(t)$ is non-decreasing continuous function whose measure is supported on $\{t : x_f(t) \in \partial D\}$. That is, $|\ell_f|(t) = \int_0^t \mathbb{1}(x_f(s) \in \partial D)\mathrm{d}|\ell_f|(s)$. Furthermore, the map $\Gamma : C([0,T],\mathbb{R}^d) \to C([0,T],\mathbb{R}^d) \times C([0,T],\mathbb{R}^d)$ defined by

$$\Gamma(f) = \big(\Gamma_1(f), \Gamma_2(f)\big) := \big(x_f, \ell_f\big)$$

is Hölder continuous of exponent $1/2$.

tence_system}
stence_system **Theorem 2.2** (Strong Existence and (Weak) Uniqueness). *There exists a strong solution to (1).*

Remark 1. *(Strong uniqueness could possibly be proved using a similar method as Section 3 in [BBCH10]. In which case strong existence would follow from Yamada-Watanabe. In any case, strong uniqueness is not required for our results.)*

Proof. Weak uniqueness is easily proven using a Girsanov change of measure. Given our probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let

$$\mathcal{E}(t) = \exp\left\{\sum_{i=1}^n \int_0^t K^{(n)}(s)\mathrm{d}B_i(s) - \frac{n}{2}\int_0^t \left(K^{(n)}\right)^2 \mathrm{d}s\right\}$$

be the local Girsanov transform where $(X_1^{(n)}, \ldots, X_n^{(n)})$ is distributed as n i.i.d. reflected Brownian motions under the measure $\mathrm{d}\mathbb{Q}_t = \mathcal{E}(t)\mathrm{d}\mathbb{P}_t$. Because $X_i^{(n)}$ is a process reflecting inside D that satisfies

$$\big(X_i^{(n)}, L_i^{(n)}\big) = \left(\Gamma_1\Big(X_i(0) + B_i + \int_0^t K^{(n)}(s)\mathrm{d}s\Big), \Gamma_2\big(X_i(0) + B_i + \int_0^t K^{(n)}(s)\mathrm{d}s\big)\right),$$

so Hölder continuity of the local time yields

eq:Gronwall (4)
$$\begin{aligned}
\|K^{(n)}\|_{[0,t]} &\leq \frac{1}{n}\sum_{i=1}^n \|L_i^{(n)}\|_{[0,t]} = \frac{1}{n}\sum_{i=1}^n \left\|\Gamma_2\big(X_i(0) + B_i + \int_0^t K^{(n)}(s)\mathrm{d}s\big)\right\| \\
&\leq \frac{1}{n}\sum_{i=1}^n \left\|X_i(0) + B_i + \int_0^t K^{(n)}(s)\mathrm{d}s\right\|^{1/2} \\
&\leq \frac{C}{n}\sum_{i=1}^n \|X_i(0) + B_i\|_{[0,t]}^{1/2} + \frac{C}{n}\sum_{i=1}^n \left(\int_0^t \|K^{(n)}\|_{[0,s]}\mathrm{d}s\right)^{1/2} \\
&= \frac{C}{n}\sum_{i=1}^n \|X_i(0) + B_i\|_{[0,t]}^{1/2} + C\left(\int_0^t \|K^{(n)}\|_{[0,s]}\mathrm{d}s\right)^{1/2}.
\end{aligned} \tag{4}$$

Let

$$\alpha_n(t) := \frac{C}{n}\sum_{i=1}^{n} \|X_i(0) + B_i\|_{[0,t]}^{1/2},$$

and

$$\beta_n(t) := \|K^{(n)}\|_{[0,t]}.$$

Then we have

$$\beta_n(t) \le \alpha_n(t) + C\left(\int_0^t \beta_n(s)\mathrm{d}s\right)^{1/2}.$$

By a non-linear version of the Grönwall's inequality we get that

q:bound_drift (5) {eq:bound_drift}
$$\beta_n(t) \le C'(t\alpha_n(t) + t)^2, \tag{5}$$

and so $\beta_n(t)$ does not blow up in finite time. From this, and standard stopping time arguments, weak uniqueness of the law of $\mathbb{P}_t$ follows from weak-uniqueness of i.i.d. Brownian motions reflected inside D under $\mathbb{Q}_t$.

(Strong Existence). To show strong existence we construct a sequence that converges almost surely to a solution. The construction in our proof follows along the lines of Theorem 2.5 in [Bar20]. Fix $\epsilon > 0$. We will construct a collection of processes $(X_i^{(n,\epsilon)}, L^{(n,\epsilon)})$ for $1 \le i \le n$ that has a subsequence converging to a solution as ϵ tends to zero. First, let $f_i, 1 \le i \le n$ be continuous functions from $[0,\infty)$ to D. We recursively define the following functions in time-steps of size ϵ. We hide the dependence on f and n for ease of readability. Define

$$K_\epsilon(t) = 0, \text{ for } t \in [0,\epsilon), \tag{6}$$

$$(x_i^\epsilon(t), \ell_i^\epsilon) = \Gamma(f_i + K_\epsilon, t), \text{ for } t \in [(n-1)\epsilon, n\epsilon] \text{ and } 1 \le i \le n, \tag{7}$$

$$K_\epsilon(t) = K_\epsilon(n\epsilon) + \frac{1}{n}\sum_{i=1}^{n}(t - n\epsilon)\ell_i^\epsilon(n\epsilon), \text{ for } t \in [n\epsilon, (n+1)\epsilon] \text{ and } 1 \le i \le n. \tag{8}$$

Intuitively, we are updating the drift at time steps of size ϵ to be the average local-time at the previous step. From these definitions we see that

$$K_\epsilon(t) = \int_0^t L_\epsilon(s)\mathrm{d}s$$

where

:S_Existence3 (9) {eq:S_Existence3}
$$L_\epsilon(t) = \frac{1}{n}\sum_{i=1}^{n} \ell_i^\epsilon(\lfloor t/\epsilon\rfloor\epsilon). \tag{9}$$

Furthermore,

$$(x_i^\epsilon, \ell_i^\epsilon) = \Gamma\left(f_i + \int_0^t L_\epsilon(s)\mathrm{d}s\right) = \Gamma\left(f_i + \frac{1}{n}\sum_{i=1}^{n}\int_0^t \ell_i^\epsilon(\lfloor s/\epsilon\rfloor\epsilon)\mathrm{d}s\right).$$

Consequently, the same string of inequalities given in (4) can be applied to $\|L_\epsilon\|_{[0,t]}$. That is,

$$\begin{aligned}
\|K_\epsilon\|_{[0,t]} &\leq \frac{1}{n}\sum_{i=1}^{n}\|\ell_i^\epsilon\|_{[0,t]} = \frac{1}{n}\sum_{i=1}^{n}\left\|\Gamma_2\left(f_i+\int_0^t K_\epsilon(s)\mathrm{d}s\right)\right\| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left\|f_i+\int_0^t K_\epsilon(s)\mathrm{d}s\right\|^{1/2} \\
&\leq \frac{C}{n}\sum_{i=1}^{n}\|f_i\|_{[0,t]}^{1/2} + \frac{C}{n}\sum_{i=1}^{n}\left(\int_0^t \|K_\epsilon\|_{[0,s]}\mathrm{d}s\right)^{1/2} \\
&= \frac{C}{n}\sum_{i=1}^{n}\|f_i\|_{[0,t]}^{1/2} + C\left(\int_0^t \|K_\epsilon\|_{[0,s]}\mathrm{d}s\right)^{1/2}.
\end{aligned} \tag{10}$$

Ascoli_approx (10) {eq:Arzela_Ascoli_approx}

which shows that $\alpha(s) := \|K_\epsilon\|_{[0,s]}$ satisfies the inequality

$$\alpha(s) \leq \frac{C}{n}\sum_{i=1}^{n}\|f_i\|_{[0,t]}^{1/2} + C\left(\int_0^t \alpha(s)\mathrm{d}s\right)^{1/2}.$$

By a non-linear version of Grönwall's inequality, there exists a bound, uniform in ϵ, for $\|K_\epsilon\|_{[0,t]}$. Therefore the collection $\{L_\epsilon(\cdot) : \epsilon > 0\} \subset C([0,T],\mathbb{R}^d)$ satisfies the Arzela-Ascoli criterion. Hence there exists a subsequence $\epsilon_k \to 0$ and an $L \in C([0,T],\mathbb{R}^d)$ such that $L_{\epsilon_k} \to L$ uniformly on $[0,T]$. We also see that K_{ϵ_k} converges uniformly to $\int_0^\cdot L(s)\mathrm{d}s = K$. By continuity of the map Γ there exists (x_i,ℓ_i) such that

$$(x_i^{\epsilon_k},\ell_i^{\epsilon_k}) \longrightarrow (x_i,\ell_i) = \Gamma(f_i+K), \text{for each } 1 \leq i \leq n.$$

From (9)

$$L_{\epsilon_k} \longrightarrow \frac{1}{n}\sum_{i=1}^{n}\ell_i, \text{uniformly on } [0,T].$$

In conclusion, the limits $(x_i,\ell_i,K), 1 \leq i \leq n$ satisfy

$$(x_i,\ell_i) = \Gamma\Big(f_i+K\Big) \tag{11}$$

$$K(t) = \frac{1}{n}\sum_{i=1}^{n}\int_0^t \ell_i(t). \tag{12}$$

Replacing each f_i pathwise with independent Brownian motions B_i yields the strong existence of solution (1) given by (X_i,L_i). □

3. Propagation of Chaos and Hydrodynamic Limit

Propagation of chaos for an increasing system of processes means any fixed number of processes converges to a collection of independent processes as the number of particles approaches infinity. In other words, any fixed collection in the system become asymptotically independent. See [Szn84]. If the system is exchangeable, then any fixed number of processes converges to a collection of independent processes with each having the same law.

{def:POC}
def:POC **Definition 1.** *Propagation of chaos holds for a triangular array of continuous processes* $(X_i^{(n)} : 1 \le i \le n, n \in \mathbb{N})$ *if for any* $k \in \mathbb{N}$*, and fixed indices* $i_1, \dots, i_k$*, there exists independent continuous processes* $\widetilde{X}_1, \dots, \widetilde{X}_k$ *such that*

$$(X_{i_1}^{(n)}, \dots, X_{i_k}^{(n)}) \xrightarrow{d} (\widetilde{X}_1, \dots, \widetilde{X}_k).$$

If propagation of chaos holds for a triangular array, then the empirical measure $\pi^{(n)}$ of the system will converge to the probability measure concentrated on the law induced by the limiting process if the system of processes is exchangeable. This holds when we view the empirical measure

$$\pi^{(n)}(\omega) := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(\omega)}$$

as a map from Ω to $\mathcal{P}(C([0,T],\mathbb{R}^d))$, the space of probability measures on $C([0,T],\mathbb{R}^d)$ equipped with the metric of weak convergence.

{prop:POCHyd}
prop:POCHyd **Proposition 3.1** (POC implies Hydrodynamic Limit)**.** *Let* $(X_i^{(n)} : 1 \le i \le n, n \in \mathbb{N})$ *be a triangular array of continuous processes such that for each* n*,* $(X_1^{(n)}, \dots, X_n^{(n)})$ *is exchangeable. That is,* $(X_1^{(n)}, \dots, X_i^{(n)}) \overset{d}{=} (X_{\tau(1)}^{(n)}, \dots, X_{\tau(n)}^{(n)})$ *for any permutation* $\tau \in \mathcal{S}_n$*. If propagation of chaos holds for* $(X_i^{(n)} : 1 \le i \le n, n \in \mathbb{N})$ *then*

eq:Limit {eq:Limit}

$$\pi^{(n)} \xrightarrow{d} \delta_{\widetilde{X}_1}, \tag{13}$$

where $\widetilde{X}_1$ *is the limiting process described in Definition 1, and where* $\delta_{\widetilde{X}_1} \in \mathcal{P}(C([0,T],\mathbb{R}^d))$ *is the (random) measure concentrated at* $\widetilde{X}_1$*.*

{remark:weakC
mark:weakConv **Remark 2.** *The element* $\delta_{\widetilde{X}_1}$ *is a random element in the space* $\mathcal{X} := \mathcal{P}(C([0,T],\mathbb{R}^d))$ *in the sense that* $\omega \mapsto \delta_{\widetilde{X}_1(\omega)}$ *is a map from* $\Omega \to \mathcal{P}(C([0,T],\mathbb{R}^d))$*, similar to* $\pi^{(n)}$ *which we mentioned above. We equip the space* $\mathcal{X}$ *with the metric* d *of weak convergence, so* $\pi^{(n)}$ *and* $\delta_{\widetilde{X}_1}$ *are* $(\mathcal{X}, d)$ *valued random elements. The limit in (13) simply means convergence in distribution as random elements of the metric space* $(\mathcal{X}, d)$*.*

Proof Sketch of Proposition 3.1. Let $(\mathcal{X}, d)$ be the space of measures on $C([0,T],\mathbb{R}^d)$ with the metric of weak convergence, as given in Remark 2. For a distribution Q on $\mathcal{X}$, consider Q acting on a bounded function $F : C([0,T],\mathbb{R}^d) \to \mathbb{R}$ by

$$\langle Q, F\rangle := \mathbb{E}_Q\left(\int F(\omega)\mathrm{d}\mu(\omega)\right) = \int_{\mathcal{X}}\int F(\omega)\mathrm{d}\mu(\omega)\,\mathrm{d}Q(\mu).$$

For a given F, exchangeability yields

$$\begin{aligned} &\langle \pi^{(n)} - \delta_{\widetilde{X}_1}, F\rangle^2 \\ &\quad= \frac{1}{n^2}\mathbb{E}F(X_1) + \frac{n-1}{n}\mathbb{E}\big(F(X_1^{(n)})F(X_2^{(n)})\big) - 2\mathbb{E}F(X_1^{(n)})\mathbb{E}F(\widetilde{X}_1) + [\mathbb{E}F(\widetilde{X}_1)]^2. \end{aligned} \tag{14}$$

By the propagation of chaos $(X_1^{(n)}, X_2^{(n)}) \to (\widetilde{X}_1, \widetilde{X}_2)$ where $\widetilde{X}_i, i = 1, 2$, are independent and by exchangeability have the same law. Hence $\frac{n-1}{n}\mathbb{E}\big(F(X_1^{(n)})F(X_2^{(n)})\big) \to [\mathbb{E}F(\widetilde{X}_1)]^2$, and consequently

$$\langle \pi^{(n)} - \delta_{\widetilde{X}_1}, F\rangle^2 \longrightarrow 0.$$

This says that $\pi^{(n)}$ converges to the constant □

There is another natural way of viewing $\pi^{(n)}$. For fixed t, clearly

$$\pi_t^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(t)}$$

is a random measure with support in D. That is, $\pi_t^{(n)}$ is a random element of $\mathcal{P}(D)$ equipped with metric of weak convergence. Letting t vary within the interval $[0, T]$, continuity of the processes $X_i^{(n)}(\cdot)$ implies that $\{\pi_t^{(n)} : t \in [0, T]\}$ is almost surely a continuous measure valued process (i.e. a.s. an element in $C([0, T], \mathcal{P}(D))$). We call this scheme (B). One can often show that convergence of the empirical measure in (A) implies convergence of the empirical process in (B) to the density of the limiting process in the propagation of chaos. In this section we prove a strong propagation of chaos result described below.

{th:POC}
th:POC **Theorem 3.2** (Propagation of Chaos)**.** *Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space supporting a sequence of i.i.d. $\mathbb{R}^d$-Brownian motions $B_i, i \in \mathbb{N}$. As given by Theorem 2.2 there exists a triangular array of strong solutions to (1) given by $(X_i^{(n)} : 1 \leq i \leq n, n \in \mathbb{N})$. For any fixed k,*

$$(X_1^{(n)}, \ldots, X_k^{(n)}) \longrightarrow (\widetilde{X}_1, \ldots, \widetilde{X}_k),$$

almost surely on $C([0, T], D)$, where $\widetilde{X}_i$ are independent and identically distributed. Furthermore,

sde:limit
$$\begin{aligned} \widetilde{X}_1(t) &= X_1(0) + B_1(t) + \widetilde{L}_1(t) + \int_0^t \mathbb{E}\widetilde{L}_1(s)\mathrm{d}s, \\ \widetilde{L}_1(t) &= \int_0^t \eta(\widetilde{X}_1(s))\mathrm{d}|\widetilde{L}_1|s, \end{aligned} \tag{15}$$

and $|\widetilde{L}_1|$ is the local time of $\widetilde{X}_1$ on ∂D.

The proof of Theorem 3.2 will use two main ingredients: a strong exchangeability condition for the (common) drift of the X_i's, and strong uniqueness of the solution to the limiting process (15). Essentially,

(Strong Exchangeability) + (Strong Uniqueness of Limit) $\implies$ (Propagation Of Chaos).

Definition 2. *Let $\{Y_i : 1 \leq i \leq n\}$ be a collection of random variables on a probability space $(\Omega, \mathbb{P})$. A random variable Z adapted to $\sigma(Y_1, \ldots, Y_n)$ is* strongly exchangeable *if there is a measurable function F such that $Z = F(Y_1, \ldots, Y_n)$, and $F(x_1, \ldots, x_n) = F(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any permutation $\sigma \in \mathcal{S}_n$.*

One corollary of the construction of strong existence for the system (1) given in Theorem 2.2 is that there exists a strong solution of the processes $(x_i^{(n)}, L_i^{(n)}, K^{(n)})$ where $K^{(n)}$ is strongly exchangeable with respect to the Brownian motions $(B_1, \ldots, B_n)$.

Corollary 3.3. *Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space supporting a sequence of independent d-dimensional Brownian motions $B_i, i \in \mathbb{N}$, and $((X_i^{(n)}, L_i^{(n)}) : 1 \leq i \leq n, n \in \mathbb{N})$ be the triangular array on the same space as given in Theorem 2.2. Then $K^{(n)}$ is strongly exchangeable with respect to the Brownian motions $(B_1, \ldots, B_n)$ for every $n \in \mathbb{N}$.*

Prop:SDE_EU **Proposition 3.4** (Strong Existence and Uniqueness of Limiting Process)**.** *Strong existence and uniqueness holds for solutions to the system* {Prop:SDE_EU}

eq:SDE_Limit1 (16) {eq:SDE_Limit1}
$$X(t) = B(t) + L(t) + \int_0^t \mathbb{E}L(s)\mathrm{d}s,$$

eq:SDE_Limit2 (17) {eq:SDE_Limit2}
$$L(t) = \int_0^t \eta(X(s))\mathrm{d}|L|(s),$$

(15) *where $t \to |L(t)|$ is the local time of X on ∂D.*

Remark 3. *Our proof is similar to, and based on, the proof of path-wise uniqueness of Brownian motion with inert drift given in* [BBCH10].

We now prove the propagation of chaos given the above results.

Proof. We first prove pathwise uniqueness. ∎

Notice that (5) in the proof of Theorem 2.2 gives a uniform (in terms of n) upper bound on the drift term $K^{(n)}$, which we state as a separate lemma below.

uniform_bound **Lemma 3.5** (Uniform bound on drift)**.** *There is a constant $C' > 0$ such that for all n,* {lemma:uniform

a_bound_drift (18) {eq:lemma_bound_drift}
$$\|K^{(n)}\|_{[0,t]} \leq C' \left(\frac{t}{n} \sum_{i=1}^n \|X_i(0) + B_i\|_{[0,t]}^{1/2} + t \right)^2.$$

Proof of Theorem 3.2 (Propagation of Chaos). Let $(\Omega, (\mathcal{F}_t), \mathbb{P})$ be a probability space supporting a sequence of independent standard d-dimensional Brownian motions $B_i, i \in \mathbb{N}$. We first analyze $K^{(n)} \in \mathbb{R}^d$ by breaking it up into its components

$$K^{(n)} = \sum_{i=1}^d K_i^{(n)} e_i$$

where the $\{e_i : i = 1, \ldots, d\}$ is the collection of standard basis vectors. We show each $K_i^{(n)}$ converges to a deterministic function by showing

$$\limsup_{n\to\infty} K_i^{(n)} = \liminf_{n\to\infty} K_i^{(n)}$$

for each i, almost surely, by demonstrating that lim sup and lim inf have the same almost sure limit by strong exchangeability and the Hewitt-Savage zero-one law. We first treat the case of limit superior and $i = 1$. Let N_k be a random subsequence such that

eq:N_k

$$\lim_{k\to\infty} K_1^{(N_k^1)} = \limsup_{n\to\infty} K_1^{(n)} =: \overline{K}_1 \tag{19} \quad \{eq:N\_k\}$$

almost surely. We can choose N_k^1 to be strongly exchangeable because the sequence $K_1^{(n)}$ is strongly exchangeable. In other words, for a given $\omega = (\omega_1, \omega_2, \dots)$ of the Brownian paths $(B_1, B_2, \dots)$ we know $K_1^{(n)}(\omega_\sigma) = K_1^{(n)}(\omega)$ for large enough n, a.s., where $\omega_\sigma := (\omega_{\sigma(1)}, \omega_{\sigma(2)}, \dots)$ is the sequence of paths with indices permuted by a given permutation σ. Hence we can take $N_k^1(\omega_\sigma) = N_k^1(\omega)$ a.s. for all such permutations σ on finite indices.

In a likewise manner we can find a strongly exchangeable subsequence of N_k^1, call it N_k^2 such that

$$\lim_{k\to\infty} K_2^{(N_k^2)} = \limsup_{k\to\infty} K_2^{(N_k^1)},$$

almost surely. Inducting over the indices of the coordinates, we can repeat this to find strongly exchangeable subsequences $N_k^1, N_k^2, \dots, N_k^d$ such that (19) holds, and in addition

$$\lim_{k\to\infty} K_i^{(N_k^i)} = \limsup_{k\to\infty} K_i^{(N_k^{i-1})}$$

for $1 < i \leq d$. Furthermore, the last subsequence N_k^d has the property that $K_i^{(N_k)}$ has an almost sure limit for each $i = 1, \dots, d$. In particular, the limit of the first coordinate converges to its original limit superior. That is,

$$\limsup_{k\to\infty} K_1^{(N_k^d)} = \overline{K}_1,$$

almost surely. Define

$$\overline{K} = \lim_{k\to\infty} K^{(N_k^d)} = \lim_{k\to\infty} \sum_{i=1}^{d} K_i^{(N_k^d)} e_i,$$

and notice $\overline{K}$ is strongly exchangeable with respect to the sequence of i.i.d. Brownian motions $B_{1,2}, \dots$ because $K^{(n)}$ and N_k^d are both strongly exchangeable. By the Hewitt-Savage zero-one law, $\overline{K}$ is in fact a deterministic function. By Lemma 3.5 we see each coordinate of $K^{(N_k)}$ is uniformly bounded in k, almost surely, and consequently the bounded convergence theorem implies

$$\left\| \int_0^\cdot K^{(N_k^d)}(s)\mathrm{d}s - \int_0^\cdot \overline{K}(s)\mathrm{d}s \right\|_{[0,t]} \longrightarrow 0,$$

almost surely, as $k \to \infty$.

In other words, the drift $\int_0^\cdot K^{(N_k^d)}(s)\mathrm{d}s$ converges a.s. to the (deterministic) continuous function $\int_0^\cdot \overline{K}(s)\mathrm{d}s$. We now couple a collection of reflected process with this deterministic drift. For each i, define

$$\widetilde{X}_i := \Gamma_1\left(X_i(0) + B_i + \int_0^{\cdot} \overline{K}(s)\mathrm{d}s\right), \tag{20}$$

so that

$$\widetilde{X}_i(t) = X_i(0) + B_i(t) + \int_0^t \overline{K}(s)\mathrm{d}s + \widetilde{L}_i(t)$$

where

eq:X_tilde

$$\widetilde{L}_i = \Gamma_2\left(X_i(0) + B_i + \int_0^{\cdot} \overline{K}(s)\mathrm{d}s\right) \tag{21}$$

{eq:X_tilde}

is the reflection local time of $\widetilde{X}_i$ on ∂D. So $\widetilde{X}_i$ is an approximation of $X_i^{(N_k^d)}$ where the drift of $X_i^{(N_k^d)}$ approaches the drift of $\widetilde{X}_i$ (for each i). By Hölder continuity of Γ, we have

eq:LT_approx

(22) {eq:LT_approx}

$$\begin{aligned}
&\left\| K^{(N_k^d)} - \frac{1}{N_k^d}\sum_{i=1}^{N_k^d} \widetilde{L}_i \right\|_{[0,t]} \leq \frac{1}{N_k^d}\sum_{i=1}^{N_k^d} \|\widetilde{L}_i - L_i^{(N_k^d)}\|_{[0,t]} \\
&= \frac{1}{N_k^d}\sum_{i=1}^{N_k^d} \left\| \Gamma_2\left(X_i(0) + B_i + \int_0^{\cdot} \overline{K}(s)\mathrm{d}s\right) - \Gamma_2\left(X_i(0) + B_i + \int_0^{\cdot} K^{(N_k^d)}\mathrm{d}s\right) \right\|_{[0,t]} \\
&\leq \frac{C}{N_k^d}\sum_{i=1}^{N_k^d} \left\| \int_0^{\cdot} \overline{K}(s)\mathrm{d}s - \int_0^{\cdot} K^{(N_k^d)}(s)\mathrm{d}s \right\|_{[0,t]}^{1/2} \\
&\longrightarrow 0,
\end{aligned}$$

almost surely. On the other hand, it is clear that $(\widetilde{X}_i : i \in \mathbb{N})$ is a collection of independent processes because $\widetilde{X}_i$ is a function of B_i and $\overline{K}$, and $\overline{K}$ is deterministic. Likewise, $(\widetilde{L}_i : i \in \mathbb{N})$ is an independent collection of processes. The strong law of large numbers implies that

$$\frac{1}{n}\sum_{i=1}^{n} \widetilde{L}_i(t) \longrightarrow \mathbb{E}\widetilde{L}_1(t),$$

almost surely. It follows from (22) that

$$\int_0^{\cdot} \overline{K}(s)\mathrm{d}s = \mathbb{E}\widetilde{L}_1(t),$$

almost surely, as well. This means the equation (21) for $\widetilde{X}_i$ becomes

$$\widetilde{X}_i(t) = X_i(0) + B_i(t) + \widetilde{L}_i(t) + \int_0^t \mathbb{E}\widetilde{L}_i(s)\,\mathrm{d}s.$$

By Proposition 3.4 we have (strong) uniqueness of the above equation, and in particular we know that $\mathbb{E}\widetilde{L}_i(t)$ is a well defined continuous function in $\mathbb{R}^d$. In other words, looking back at our original definition of $\overline{K}$, we see

$$\mathbb{E}\widetilde{L}_1 = \overline{K} = \lim_{k\to\infty} K^{(N_k^d)}.$$

In particular, the first coordinate function of $\overline{K}$ is the first coordinate function of G. That is,

$$\pi_1\big(\mathbb{E}\widetilde{L}_1(t)\big) = \overline{K}_1(t) = \limsup_{n\to\infty} K_1^{(n)}(t),$$

for all $t \in [0, T]$, almost surely, where $\pi_i : \mathbb{R}^d \to \mathbb{R}$ is the projection onto the ith coordinate.

The argument until this point can be repeated with the limit superior replaced by limit inferior in the definition of $\overline{K}$, demonstrating

$$\liminf_{n\to\infty} K_1^{(n)} = \pi_1\mathbb{E}\widetilde{L}_1(t),$$

almost surely. That is, the limit superior and limit inferior of the first coordinate agree almost surely, hence

$$\lim_{n\to\infty} K_1^{(n)} = \pi_1\mathbb{E}\widetilde{L}_1(t)$$

almost surely. This shows the existence and uniqueness of $K_1^{(n)}$, the first coordinate of $K^{(n)}$, and inducting over the other coordinates we see

$$\lim_{n\to\infty} K_i^{(n)} = \pi_i\mathbb{E}\widetilde{L}_1(t),$$

almost surely. Combining this together, we see that

$$\lim_{n\to\infty} K^{(n)} = \mathbb{E}\widetilde{L}_1(t),$$

almost surely, where $\mathbb{E}\widetilde{L}_1(t)$ is the expected local time of the solution given in the (15). This implies that

$$\begin{aligned} X_i^{(n)} &= \Gamma_1\left(X_i(0) + B_i + \int_0^\cdot K^{(n)}(s)\mathrm{d}s\right) \\ &\xrightarrow{a.s.} \Gamma_1\left(X_i(0) + B_i + \int_0^\cdot \mathbb{E}\widetilde{L}_1(s)\mathrm{d}s\right) \\ &= \widetilde{X}_i. \end{aligned}$$

This completes the proof of the propagation of chaos.

4. Hydrodynamic Limit: Empirical Measure vs. Empirical Process

The empirical process converges to the probability measure concentrated on the law induced by the limiting process given in the propagation of chaos provided the system of processes is exchangeable, see Sznitman [Szn84]. In this way we are viewing the empirical measure

$$\pi^{(n)} := \frac{1}{n}\sum_{i=1}^n \delta_{X_i^{(n)}}$$

as a map from Ω to $\mathcal{P}(C([0,T],\mathbb{R}^d))$, the space of probability measures on $C([0,T],\mathbb{R}^d)$ equipped with the metric of weak convergence. That is, δ_f is the delta mass concentrated on the continuous function f. We call this view of $\pi^{(n)}$ scheme (A), and we will call $\pi^{(n)}$ the *empirical measure*.

There is another natural way of viewing $\pi^{(n)}$, and that is viewing it as a measure-valued process. For fixed t, clearly

$$\pi_t^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(t)}$$

is a random measure with support in D. Here δ_x is delta mass concentrated at the location $x \in \mathbb{R}^d$. That is, $\pi_t^{(n)}$ is a random element of $\mathcal{P}(D)$ equipped with metric of weak convergence. Letting t vary within the interval $[0,T]$, continuity of the processes $X_i^{(n)}(\cdot)$ demonstrate that $\{\pi_t^{(n)} : t \in [0,T]\}$ is almost surely a continuous measure valued process (i.e. a.s. an element in $C([0,T],\mathcal{P}(D))$). We call this scheme (B), and will call $\pi^{(n)}(\cdot)$ the *empirical process*. One can often show that convergence of the empirical measure in (A) implies convergence of the empirical process in (B) to the density of the limiting process in the propagation of chaos.

Theorem 4.1 (HL, Scheme (A))**.** *Assume the same setting as in Theorem 3.2 (Propagation of Chaos), and let $\widetilde{X}_1$ be the limiting process in that theorem statement. Consider the empirical measure $\pi^{(n)}(\omega) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(\omega)}$. Then,*

$$\pi^{(n)}$$

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